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# Accuracy barriers in mesh adaptation

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**Abstract:** The convergence rate of the approximation of a non-smooth, typically discontinuous function can be made higher than usual when an adaptative mesh refinement strategy is applied (in contrast to the uniform mesh refinement). But, for isotropic mesh refinement, this order can be limited to $d/p(d-1)$ for the $L^p$ norm and a space dimension $d$.

**Key-words:** mesh, adaption, approximation, interpolation

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# Barrières de précision en maillage adaptatif

**Résumé :** Le taux de convergence pour l'approximation d'une fonction discontinue ou irrégulière peut présenter un ordre plus élevé grâce à l'utilisation d'une stratégie de raffinement adaptatif, à la place d'une méthode de raffinement uniforme. Dans tous les cas, cet ordre de convergence est cependant limité à $d/p(d-1)$ (en norme $L^p$ et en dimension $d$) pour les adaptations isotropes.

**Mots-clés :** maillage, adaptation, approximation, interpolation

# 1 Introduction

In spite of the extraordinary progress in efficiency of the combination of numerical methods and computers, most of industrial flow simulations are performed within very poor accuracy conditions. The maximal theoretical order of accuracy is often less than two. Even when it is two, this is not observed on the results, due to the lack of smoothness of the flow fields under study. Either these flow fields are rigourously singular (inviscid shocks, singularities at angular walls), or they are regular, but with spatial scales too small for being captured with the theoretical order of accuracy. In both cases, the apparent order of accuracy can be one or even lower.

We note in passing that our remark applies also to the so-called high resolution schemes, involving limiters, which indeed proved their ability in capturing many singularities in a thin subregion of the mesh and without Gibbs phenomenon. They however do not show a better acuracy that first-order on discontinuities (for any usual measurable functions norm).

One way to escape from the limitation of accuracy on singularities is to adopt a mesh adaptation approach.

In [1], the notion of accuracy of adaptative methods is discussed. It is shown that some of these methods are able to capture 1D discontinuities with orders of accuracy equal to two or better. This means that a global mesh adaptative method, combining an approximation scheme and a well defined adaptative remeshing, possesses its own order of accuracy for some family of discontinuities. Identifying this order of accuracy will rely on two analyses:

- from one side, it is necessary to analyse the accuracy in the regular case, that is the accuracy order of the approximation scheme, on a non-uniform mesh;
- from the other side, we also need to analyse the mesh adaptation strategy. We can interprete this strategy as a system control one. A sensor or captor will detect the region of mesh to refine or coarsen, an actuator will build the new mesh.

Most of published works in the literature examine the captor, i.e. the error estimate. In our work, we assume we have a perfect captor. We focus on the issue that even under this assumption, many remeshing strategies carries with them a strong limitation of the order of accuracy of the global adaptative approach.

More precisely, the main *barrier*, the existence of which we want establish, is described by the following lemma:

***Lemma 1.1***
*If a $\mathcal{P}_1$ interpolation method is applied to a piecewise regular function $u$, defined on $\mathbb{R}^d$, and involving discontinuities, then the convergence order $\alpha$ in $L^p$ $(1 \leq p < \infty)$ when mesh is refined with an adaptative isotropic mode can be not better than*

$$\alpha \leq \frac{1}{p}\frac{d}{d-1}.$$

Such barriers are rather simple to proof and of general application. In particular we do not consider a particular PDE. Indeed, the limitation put in evidence for interpolation should extend to PDE approximated solutions.

The information delivered is also a simple one: 3D isotropic mesh adaption such as AMR((*Adaptive Mesh refinement*, in which cubic volumes are locally divided in identical smaller cubic ones) are not second-order accurate, i.e. eight times more volumes do not yield four times smaller error.

In Section 2 we recall how the accuracy order of a mesh adaptive (or not) method is defined. In Section 3, we establish the limiting order of accuracy for isotropic mesh adaptive methods, for different kind of singularities in $\mathbb{R}^d$,for $d = 1, 2, 3$. The last section illustrates these barriers with a modern mesh adaptive tool.

# 2 Meshes and complexity

We consider the interpolation of more or less smooth functions defined in a regular bounded domain $\Omega$ of $\mathbb{R}^d$ ($d = 1, 2$, ou $3$).

The interpolation methods under study are built from a discretization of the computational domain $\Omega$, the mesh, $T$, that relies on a partition of $\Omega$ into $N$ polygonal elements, that we denote $K$, and that are assumed to be convex, open sets, such that $\cup_T \bar{K} = \bar{\Omega}$.

We examine the convergence properties of an interpolation method over a sequence of meshes $(T_p)_{p\in\mathbb{N}}$. We assume that in this sequence of meshes, the size of elements tends to zero. Accordingly the number of elements tends to infinity.

***Definition 2.1 (Diameter and roundness)***
*For any polygon $K$ in $\mathbb{R}^d$, its diameter $h_K$ is the diameter of the smallest sphere containing it and its roundness $\rho_K$ is the diameter of the largest sphere inside it.*

The ratio $\rho_K/h_K$ measures the shape of $K$ : when it is close to 1, $K$ is close to a sphere or nearly isotropic, when it is close to 0, $K$ is much stretched or strongly anisotropic.

***Definition 2.2 (Quasi-uniform meshes sequences)***
*A sequence of meshes $(T_p)_{p\in\mathbb{N}}$ is quasi-uniform if*

$$\exists\, c > 0, \quad \forall p \in \mathbb{N}, \quad c \max_{T_p} h_K \le \min_{T_p} \rho_K .$$

*Then*

$$h = \max_{T_p} h_K$$

*is a measure of the mesh size for any element of the sequence.*

In a sequence of uniform meshes, no element has a roundness smaller than $c\,h$. The elements are essentially of same size and roundness. The global mesh fineness or size is defined roughly by a single number. In other words, the number of elements $N_p$ is related to the size $h_p$ :

$$\exists\, c^\star > 0, \quad \forall p \in \mathbb{N}, \quad c\, h_p^d \,\leq\, c^\star \frac{\text{meas}\,(\Omega)}{N_p} \,\leq\, h_p^d, \tag{1}$$

since $\rho_K^d \leq c^\star \text{meas}\,(K) \leq h_K^d$ for any $K$ ($c$ is the constant arising in Definition 2.2). This can be also written:

$$h_p \,=\, O\left(\left(\frac{1}{N_p}\right)^{1/d}\right).$$

***Definition 2.3 (Mesh isotropy)***
*A sequence of meshes $(T_p)_{p\in\mathbb{N}}$ is isotropic if it satisfies*

$$\exists\, c > 0, \quad \forall p \in \mathbb{N}, \quad c \,\leq\, \min_{T_p} \frac{\rho_K}{h_K}.$$

This property ensures against an infinite stretching i.e. the ratio $\rho_K/h_K$ cannot tend to zero 0. However, in contrast to uniform meshes, in an arbitrary isotropic mesh the elements can be of very different size.

We restrict now to meshes involving simplexes, i.e. in 2D, triangles, and in 3D, tetrahedra.

# 3 Accuracy of an interpolation

In the standard theory, the approximation error is measured as a function of local mesh size. This notion is well adapted to uniform sequences of meshes. It extends also to sequences that are uniformly refined.
In mesh adaption methods the meshes are not uniform, and the sequences are not uniformly refined.
In the classical approximation theory, the mesh fineness is a measure of the cost to be paid in order to compute on the considered mesh. The idea of considering for adapted meshes simply the total number of degrees of freedom is a natural extension that is getting progressively more popularity.

## 3.1 Interpolation method

We focus on the interpolation of functions of $H^k(\Omega) \cap L^\infty(\Omega)$ with $k \in \mathbb{N}$. An interpolation method is a mapping which defines from any mesh $T$ of an open set $\Omega$ in $\mathbb{R}^d$, an operator $\pi_T$ such that

$$\pi_T \,:\, u \in H^k(\Omega) \cap L^\infty(\Omega) \longrightarrow \pi_T u \in L^\infty(\Omega).$$

Let us consider the interpolatd functions over our sequence of meshes: $u_p = \pi_{T_p} u$. It is natural to introduce the following property:

$$\exists\ c > 0, \quad \forall p \in \mathbb{N},\ \forall u \in H^k(\Omega) \cap L^\infty(\Omega)$$
$$\|u_p\|_{L^\infty(\Omega)} \leq c \min(\|u\|_{H^k(\Omega)}, \|u\|_{L^\infty(\Omega)}), \qquad \|u_p - u\|_{L^2(\Omega)} \underset{p \longrightarrow +\infty}{\longrightarrow} 0. \quad (2)$$

Two classical examples satisfy this property:

- the Lagrange interpolation on simplectic meshes [8] ;
- the $L^2$-orthogonal and piecewise constant projection (for any kind of meshes).

The first example is a usual one in finite element theory, while the second one is adapted to finite volume methods.

## 3.2 Accuracy order

***Definition 3.1***
*An interpolation method has an order of accuracy of (at least) $\alpha$ for a sequence of meshes $(T_p)_{p\in\mathbb{N}}$ if*

$$\exists\ c > 0, \quad \forall u \in H^k(\Omega) \cap L^\infty(\Omega),\ \forall p \in \mathbb{N}, \quad \|u - u_p\|_{L^2(\Omega)} \leq c \left(\frac{1}{N_p}\right)^{\alpha/d},$$

*where $N_p$ holds for the number of elements of $T_p$.*

From statement (1), this definition is coherent with the classical one for uniform meshes or uniform refinement.

## 3.3 Some examples with quasi-uniform meshes

In the classical theory, the two key assumptions in the accuracy statement of an interpolation are :

- the regularity of meshes ;
- the local (at element level) accuracy of the interpolation ;

The following fundamental result (see [8]) is a consequence of the Bramble-Hilbert lemma:

***Theorem 3.1 (Interpolation fundamental property)***
*If $\pi$, an interpolation operator on $K$,* is exact for any polynomial of degree less or equal to *$k \geq 0$, if $m \in \mathbb{N}$, then there exists a constant $c > 0$ independent of $K$ such that*

$$\forall u \in H^{k+1}(K), \quad |u - \pi u|_{m,K} \leq c\,\|B\|^{k+1}\|B^{-1}\|^m\,|u|_{k+1,K},$$

*where $B$ is a matrix such that the variable change $x = B\hat{x} + b$ transforms an element $\hat{K}$ into $K$. Here, $|\cdot|_{s,\omega}$ denotes the semi-norm of $H^s(\omega)$.*

This result applies in particular to simplectic meshes which are all images of an unique simplex, the so-called reference element $\hat{K}$. We also remark that:

$$\frac{\rho_K}{\hat{\rho}} \leq \|B\| \leq \frac{h_K}{\hat{\rho}}, \quad \frac{\hat{\rho}}{\rho_K} \leq \|B^{-1}\| \leq \frac{\hat{h}}{\rho_K}.$$

Then the following statement will hold

$$\|B\|^{k+1}\|B^{-1}\|^m \leq c \left(\frac{h_K}{\rho_K}\right)^m {h_K}^{k+1-m} \leq c\, {h_K}^{k+1-m},$$

in as much as the meshes verify the isotropy property 2.3. However a conclusion concerning the accuracy order will not be obtained unless one can relate $h_K$ to the number of elements, which is more easily done with uniform meshes.

### $H^{k+1}(\Omega)$ Functions

For rather regular functions, at least in $H^1(\Omega)$, and for quasi-uniform meshes, the following result is derived:

***Theorem 3.2***
*Let $(\pi_p)_{p\in\mathbb{N}}$ be a family of interpolation operators that are exact for any polynomial function of degree less or equal to $k$ on the simplectic elements of the meshes $(T_p)_{p\in\mathbb{N}}$, then if the* sequence of meshes $(T_p)_{p\in\mathbb{N}}$ is uniform *according to Definition 2.2, we have the following property*

$$\forall u \in H^{k+1},\ \forall p \in \mathbb{N}, \quad |u - u_p|_{m,\Omega} \leq c\,|u|_{k+1,\Omega}\, h_p^{k+1-m} \leq c\,|u|_{k+1,\Omega} \left(\frac{1}{N_p}\right)^{(k+1-m)/d}.$$

For a more general result (in $W^{m,p}(\Omega)$), see [4].

### $L^2(\Omega)$ Functions

***Theorem 3.3***
*Let $(\pi_p)_{p\in\mathbb{N}}$ be a family of interpolation operators satisfying the stability property (2), on a* sequence of meshes $(T_p)_{p\in\mathbb{N}}$, *then*

$$\forall u \in L^2(\Omega)\cap L^\infty(\Omega),\ \forall p \in \mathbb{N}, \quad \|u - u_p\|_{L^2(\Omega)} \leq c\|u\|_{L^\infty(\Omega)}\, meas\,(\Omega)^{1/2}.$$

This results is not directly useful (applied in $\Omega$). But it can be combined with Theorem 3.1 for obtaining the convergence order for functions that are piecewise $H^{k+1}$:

- in regularity regions, Theorem 3.1 can be applied,
- in non-regularity regions, the above result is applied, but replacing $\Omega$ by a neighborhood of the singularity the measure of which tends to zero.

### A first example: $P_1$-exact interpolation

Let us consider the Lagrange interpolation of degree 1 for continuous functions defined on $\bar{\Omega}$ and for simplectic meshes.

- For functions of $H^2(\Omega)$, which are in particular continuous in 1D and 2D, and possibly discontinuous at an insulated point in 3D, the second order convergence holds:

$$\forall u \in H^2, \quad \|u - u_p\|_{L^2(\Omega)} \leq c \, \|u\|_{H^2(\Omega)} \left(\frac{1}{N_p}\right)^{2/d}.$$

- For function of $H^1(\Omega)$, which are in particular continuous in 1D, possibly discontinuous on points in 2D, possibly discontinous on curves in 3D, only first order convergence holds:

$$\forall u \in H^1, \quad \|u - u_p\|_{L^2(\Omega)} \leq c \, \|u\|_{H^1(\Omega)} \left(\frac{1}{N_p}\right)^{1/d}.$$

### A second example : $P_0$-exact interpolation

We consider now the $L^2$-orthogonal projection onto the elementwise constant functions.

$$\forall u \in L^2(\Omega), \quad \pi_p u(x) = \frac{1}{\text{meas }(K)} \int_K u(x)\, dx \quad (\forall x \in K, \ \forall K \in T_p),$$

on mesh $T_p$. By application of Theorem 3.1, the accuracy statement writes as follows:

- For $H^1(\Omega)$ functions, first order convergence holds:

$$\forall u \in H^1, \quad \|u - u_p\|_{L^2(\Omega)} \leq c \, \|u\|_{H^1(\Omega)} \left(\frac{1}{N_p}\right)^{1/d},$$

  on uniform simplectic meshes (Definition 2.2)

### A third example: piecewise $H^1$ function

The functions that we consider now are functions of $u \in L^2(\Omega) \cap L^\infty(\Omega)$ that are piecewise $H^1$ in $\Omega$. A typical example in 1D, i.e. for $\Omega = ]0,1[$, $\Omega^- = ]0, 0.5[$, $\Omega^+ = ]0.5, 1[$, is

$$u(x) = \begin{cases} x & \text{si } x \in \Omega^- \\ x - 0.5 & \text{si } x \in \Omega^+, \end{cases}$$

which is a function from $\Omega = ]0,1[$ to $\mathbb{R}$, belonging to $H^1(\Omega^-)$ and to $H^1(\Omega^+)$, but being discontinuous at $x = 0.5$ ($u \notin H^1(\Omega)$). Uniform meshes $(T_p)_{p\in\mathbb{N}}$ according to Definition 2.2 are

$$T_p^0 = \{K \in T_p \ ; \ 0.5 \in K\},$$
$$T_p^- = \left\{K \in T_p \ ; \ K \subset \Omega^-\right\} \setminus T_p^0, \quad T_p^+ = \left\{K \in T_p \ ; \ K \subset \Omega^+\right\} \setminus T_p^0.$$

We define three open sets $\Omega_p^0$, $\Omega_p^-$, $\Omega_p^+$ :

$$\overline{\Omega_p^0} = \bigcup_{T_p^0} \bar{K}, \quad \overline{\Omega_p^-} = \bigcup_{T_p^-} \bar{K}, \quad \overline{\Omega_p^+} = \bigcup_{T_p^+} \bar{K}.$$

Then we can write:

$$\|u - \pi_p u\|_{L^2(\Omega_p^-)} \leq c\|u\|_{H^1(\Omega_p^-)} h_p \leq c\|u\|_{H^1(\Omega^-)} \left(\frac{1}{N_p}\right),$$

$$\|u - \pi_p u\|_{L^2(\Omega_p^+)} \leq c\|u\|_{H^1(\Omega_p^+)} h_p \leq c\|u\|_{H^1(\Omega^+)} \left(\frac{1}{N_p}\right),$$

$$\|u - \pi_p u\|_{L^2(\Omega_p^0)} \leq c\|u\|_{L^\infty(\Omega)} \left(\text{card}\left(T_p^0\right) h_p\right)^{1/2} \leq c\|u\|_{L^\infty(\Omega)} h_p^{1/2}.$$

since (in 1D) card $\left(T_p^0\right) \leq 1$. As a result, the interpolation method has an accuracy order of only $1/2$:

$$\|u - \pi_p u\|_{L^2(\Omega)} \leq c \max\left(\|u\|_{H^1(\Omega^-)}, \|u\|_{H^1(\Omega^+)}, \|u\|_{L^\infty(\Omega)}\right) \left(\frac{1}{N_p}\right)^{1/2}.$$

The same argument extends to any dimension $d \leq 3$ with $\Omega = ]0, 1[^d$, $\Omega^- = ]0, 0.5[\times]0, 1[^{d-1}$, $\Omega^+ = ]0.5, 1[\times]0, 1[^{d-1}$, and:

$$u(x) = \begin{cases} x_1 & \text{si } x \in \Omega^- \\ x_1 - 0.5 & \text{si } x \in \Omega^+, \end{cases}$$

where $x = (x_1, \ldots, x_d)$. $S = \{x \in \Omega \; ; \; x_1 = 0.5\}$ is the discontinuity surface of $u$. Let

$$T_p^0 = \{K \in T_p \; ; \; K \cap S \neq \emptyset\},$$

while $T_p^-$, $T_p^+$, $\Omega_p^0$, $\Omega_p^-$ et $\Omega_p^+$ are defined as above. Then the $1/2$ accuracy order also holds

$$\|u - \pi_p u\|_{L^2(\Omega)} \leq c \max\left(\|u\|_{H^1(\Omega^-)}, \|u\|_{H^1(\Omega^+)}, \|u\|_{L^\infty(\Omega)}\right) \left(\frac{1}{N_p}\right)^{1/2d},$$

since in that case, meas $\left(T_p^0\right) \leq 2h_p \leq 2\left(\frac{1}{N_p}\right)^{1/d}$.

***Remark 3.1***
*In dimension $d$, if the discontinuity has a dimension $m < d$, then the set $\Omega_p^0$ on each mesh will have a measure of the order of $h_p^{d-m}$, and the convergence order will be of $(d-m)/2$ (in the above, we have $m = d - 1$).*

***Remark 3.2***
*The mesh sequence $(T_p)_{p\in\mathbb{N}}$ may be built in order to satisfy $\Omega_p^0 = \emptyset$ $(\forall p \in \mathbb{N})$: it is enough to build them in such a way that the discontinuity point $x = 0.5$ be exactly at the boundary of two neighboring elements/intervals.*
*In that case the interpolation recover a better accuracy order. However, in practice, we usually do not know the location of the discontinuity point.*

# 4 Adaptative interpolation

We keep considering piecewise regular functions. We shall investigate how to improve the above limit by choosing now adaptative sequences of meshes, that is sequences of meshes that are designed in a manner adapted to the precise function we want to interpolate.

In a first step we shall consider sequences of isotropic (see Definition 2.3) adaptive meshes.

## 4.1 1D case

In 1D, adaptive mesh interpolation methods can be made of order $\alpha$, as large than the theoretical order of the interpolator, on functions that have isolated discontinuities, as stated in the following lemma:

***Lemma 4.1*** *$(d = 1)$*
*Let us assume that the interpolation method yields operators that are of accuracy order $\alpha$ in $H^k$ with uniform sequences of meshes.*

*Let $a = a_0 < a_1 < \ldots < a_n = b$ points of $\mathbb{R}$ and $u \in L^2(]a,\, b[) \cap L^\infty(]a,\, b[)$ a function such that*

$$u \in H^k(]a_{i-1},\, a_i[), \quad (i = 1 \ldots n), \qquad u \notin K^k(]a, b[).$$

*Then there exists a sequence of meshes $(T_p)_{p\in\mathbb{N}}$ de $\Omega = ]a,\, b[$ such that $(\pi_p)_{p\in\mathbb{N}}$ present an order of accuracy equal to $\alpha$ on $\Omega$.*

**Proof** We can assume without loss of generality that $n = 1$, $\Omega = ]0,\, 1[$, $u \in H^1(\Omega^-)$, $u \in H^1(\Omega^+)$, with $\Omega^- = ]0,\, 0.5[$ and $\Omega^+ = ]0.5,\, 1[$.

As above, if $T_p$ is a mesh of $\Omega$ with $N_p$ intervals, we can distinguish:

$$T_p^0 = \{K \in T_p \; ; \; 0.5 \in K\},$$
$$T_p^- = \left\{K \in T_p \; ; \; K \subset \Omega^-\right\} \setminus T_p^0, \quad T_p^+ = \left\{K \in T_p \; ; \; K \subset \Omega^+\right\} \setminus T_p^0,$$

and subregions $\Omega_p^0$, $\Omega_p^-$, $\Omega_p^+$ such that

$$\overline{\Omega_p^0} = \bigcup_{T_p^0} \bar{K}, \quad \overline{\Omega_p^-} = \bigcup_{T_p^-} \bar{K}, \quad \overline{\Omega_p^+} = \bigcup_{T_p^+} \bar{K}.$$

In order to reduce the error by a factor $2^\alpha$, we apply the following strategy:

- in $\Omega_p^-$ (resp. $\Omega_p^+$), the element size has to be divided by a factor 2 since in this regularity region, the theoretical accuracy order of the interpolation applies;
- on the contrary, on $\Omega_p^0$, element size has to be divided by $2^{2\alpha}$, since on this non-regularity region, the interpolation accuracy order is only 1/2.

We shall show how this strategy allows to build an adequate sequence of meshes, that is satisfying the announced accuracy statement.

We start with $T_0$, made of only one element, the whole interval $\Omega$.

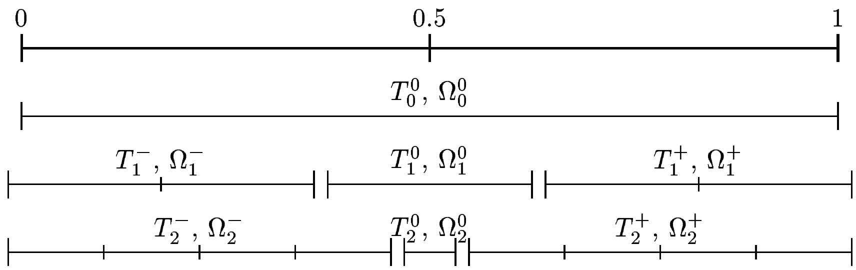


Figure 1: Computational domain $\Omega$ and meshes $T_0$, $T_1$, $T_2$

**Step 1:** in case where, $T_0^0 = T_0$, $T_0^- = \emptyset$, $T_0^+ = \emptyset$, we propose to choose:

$$T_1^0 = \{]a_1^-, a_1^+[\}$$

so that

$$\frac{\text{meas }(\Omega_1^0)}{\text{meas }(\Omega_0^0)} = \frac{1}{4^\alpha},$$

and for example

$$a_1^- = \frac{1}{2} - \frac{1}{2}\frac{1}{4^\alpha}, \quad a_1^+ = \frac{1}{2} + \frac{1}{2}\frac{1}{4^\alpha}.$$

The above implies that $\Omega_1^- = \{]0, a_1^-[\}$ and $\Omega_1^+ = \{]a_1^+, 1[\}$. It suffices now to divide $\Omega_1^-$ (resp. $\Omega_1^+$) into two intervals of equal measures. The mesh size for $\Omega_1^-$ and $\Omega_1^+$ is then uniform and equal to

$$h_1^- = h_1^+ = \frac{1}{2}\frac{1}{2}\left(1 - \frac{1}{4^\alpha}\right).$$

**Step 2:** we choose

$$T_2^0 = \{]a_2^-, a_2^+[\}$$

so that

$$\frac{\text{meas }(\Omega_2^0)}{\text{meas }(\Omega_1^0)} = \frac{1}{4^\alpha}, \quad \text{being meas }(\Omega_2^0) = \left(\frac{1}{4^\alpha}\right)^2,$$

for example

$$a_2^- = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{4^\alpha}\right)^2, \quad a_2^+ = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{4^\alpha}\right)^2.$$

We then have $\Omega_2^- = \{]0, a_2^-[\}$ and $\Omega_2^+ = \{]a_2^+, 1[\}$ and it suffices to divide $\Omega_2^-$ (resp. $\Omega_2^+$) into $2^2$ equal intervals, the mesh size of $\Omega_2^-$ and $\Omega_2^+$ is uniform and equal to

$$h_2^- = h_2^+ = \frac{1}{2^2}\frac{1}{2}\left(1 - \left(\frac{1}{4^\alpha}\right)^2\right).$$

By iterating this process, we obtain a sequence of meshes $(T_p)_{p\in\mathbb{N}}$ where

$$\text{meas}\left(\Omega_p^0\right) = \left(\frac{1}{4^\alpha}\right)^p, \text{and card}\left(T_p^0\right) = 1,$$

$$h_p = \left(\frac{1}{2}\right)^p \frac{1}{2}\left((1 - \left(\frac{1}{4^\alpha}\right)^p\right) \text{ is the uniform submeshes size for } \Omega_p^\pm,$$

$$\text{card}\left(T_p^-\right) = \text{card}\left(T_p^+\right) = 2^p.$$

By construction, $\pi_p$ has an order of accuracy equal to $\alpha$ in $\Omega^-$ and $\Omega^+$, and is stable in $L^\infty$, Then the sequence $(T_p)_{p\in\mathbb{N}}$ that we have built satisfies for $p$ large enough:

$$\|u - \pi_p u\|_{L^2(\Omega_p^\pm)} \le c\|u\|_{H^1(\Omega^\pm)} \left(\frac{1}{2}\right)^{\alpha p},$$

$$\|u - \pi_p u\|_{L^2(\Omega_p^0)} \le c\|u\|_{L^\infty(\Omega)} \left(\frac{1}{4^\alpha}\right)^{p/2} = c\|u\|_{L^\infty(\Omega)} \left(\frac{1}{2}\right)^{\alpha p},$$

thus:

$$\forall p \ge p_0, \quad \|u - \pi_p u\|_{L^2(\Omega)} \le c(u)\left(\frac{1}{N_p}\right)^\alpha,$$

since the total number of elements in mesh $T_p$ is $N_p = 2\,2^p + 1$ which is equivalent to $2\,2^p$ when $p$ grows to infinity.
When $\Omega = ]a, b[$, and for a unique point of discontinuity $a_1$, the conclusion follows from the application of the variable change $x = a + \bar{x}/(b-a)$.

For a (larger than one but) finite number of discontinuities $a_1, a_2, ..., a_n$, the conclusion is obtained by cutting $\Omega$ into $n$ intervals, each of them containing one discontinuity:

$$\Omega_1 = ]a, \frac{a_1 + a_2}{2}, \ \Omega_2 = ]\frac{a_1 + a_2}{2}, \frac{a_2 + a_3}{2}[, \ \ldots, \ \Omega_n = ]\frac{a_{n-1} + a_n}{2}, b[.$$

The sequence of meshes results of the juxtaposition of the $n$ submeshes for intervals $(\Omega_i)_{i=1\ldots n}$.
□

***Remark 4.1***
*The previous results proves that higher order accuracy can be obtained on singularities with adaptive mesh strategies, while non adaptive show a very low order.*
*This remark extends mutatis mutandis to the case of regular functions, as far as they involve steep gradients. It is true that in that case a high numerical order of convergence can be observed. But, with steep gadients, this can be observed only with extremely fine, generally not affordable, meshes. Conversely, at reasonable mesh size, the convergence shows the same low convergence as for the singular case. In [1] it is illustrated that a higher order convergence can be obtained with rather coarse adapted meshes in a similar manner to the singular case.*

## 4.2 The 2D case

We shall extend the above theory to the two-dimensional case. Again we shall establish some paramount advantages of adaptative strategies, but it will be necessary to distinguish isotropic strategies from anisotropic ones.

Given $\Omega^-$, $\Omega^+$ and $\Omega$ three connex open bounded subsets of $\mathbb{R}^2$ such that $\bar{\Omega} = \overline{\Omega^-} \cup \overline{\Omega^+}$, we consider the interpolation of a function

$$u \,:\, \Omega \longrightarrow \mathbb{R}, \qquad u \in H^k(\Omega^-),\; u \in H^k(\Omega^+),\; u \in L^\infty(\Omega).$$

We denote by

$$\Gamma \,=\, \overline{\Omega^-} \cap \overline{\Omega^+}$$

the common boundary between $\Omega^-$ and $\Omega^+$ and we assume it is a continous manifold of dimension $d-1$.

For any sequence of meshes $(T_p)_{p\in\mathbb{N}}$, we set

$$T_p^0 \,=\, \{K \in T_p \,;\, K \cap \Gamma \neq \emptyset\}\,,$$
$$T_p^- \,=\, \left\{K \in T_p \,;\, K \subset \Omega^-\right\} \setminus T_p^0, \quad T_p^+ \,=\, \left\{K \in T_p \,;\, \subset \Omega^+\right\} \setminus T_p^0,$$

and we introduce $\Omega_p^0$, $\Omega_p^-$, $\Omega_p^+$ the open set defined by:

$$\overline{\Omega_p^0} \,=\, \bigcup_{T_p^0} \bar{K}, \quad \overline{\Omega_p^-} \,=\, \bigcup_{T_p^-} \bar{K}, \quad \overline{\Omega_p^+} \,=\, \bigcup_{T_p^+} \bar{K}.$$

We restrict our study to sequences of meshes satisfying the following assumptions:

$$\exists c > 0, \quad \forall p \in \mathbb{N},\; \forall K \in T_p, \quad \frac{\rho_K}{h_K} \,\geq\, c, \tag{3}$$

$$\exists c > 0, \quad \forall p \in \mathbb{N},\; \forall K \in T_p^0, \quad \frac{\text{meas }(K \cap \Omega^-)}{\text{meas }(K)} \,\geq\, c,\; \frac{\text{meas }(K \cap \Omega^+)}{\text{meas }(K)} \,\geq\, c, \tag{4}$$

$$\exists c > 0, \quad \forall p \in \mathbb{N}, \quad \text{meas }\left(\Omega_p^0 \cap \Gamma\right) \,\geq\, c, \tag{5}$$

$$\lim_{p \to +\infty} h_p \,=\, 0 \tag{6}$$

with

$$h_p = \max_{K \in T_p} \operatorname{diam}(K).$$

These assumptions express in more details the isotropy of the mesh sequence.

***Lemma 4.2** (d = 2)*
*(i) The interpolation obtained by the $L^2$-orthogonal piecewise constant projection is first order accurate for $H^1$ functions.*
*(ii) For the function*

$$u(x,\, y) = \begin{cases} 0 & si\ x < 0.5 \\ 1 & si\ x > 0.5 \end{cases} \quad \textit{defined in } \Omega = ]0,\, 1[\times]0,\, 1[$$

*any sequence of meshes $(T_p)_{p\in\mathbb{N}}$ satisfying the isotropy property 2.3 will produce an error such that*

$$c\left(\frac{1}{N_p}\right)^{1/2} \leq \|u - u_p\|_{L^2(\Omega)}.$$

*or in other words the interpolation order cannot be better than* 1.

It is difficult to produce a rigorous demonstration a result, tending to say that "no method" of a large family will be able to answer accurately to the problem of fast convergence to the seeked function.
We shall however give an argumentation relying on a counterexample that will make more precise the large class of mesh strategies which obey our result.

The idea is that the error will soon be concentrated along the discontinuity, included in the region $\Omega_p^0$ of measure $h_p^0$ containing $N_p^0 = 1/h_p^0$ elements. The error is there of order $\left(\text{meas }(\Omega_p^0)\right)^{1/2}$ i.e. $\left(1/N_p^0\right)^{1/2}$.

The counterexample which we propose has a discontinuity along a straight line of $\mathbb{R}^2$ :

$$u(x) = 0 \quad \forall \quad x < 0, \quad u(x) = 1, \quad \forall \quad x > 0 \tag{7}$$

We shall show that isotropic mesh refinements cannot allow an order of convergence for $\mathcal{P}_1$ interpolations of function $u$ that could be better than 1.

For a given triangulation, since $u$ is constant far from its discontinuity line, interpolation errors are there vanishing and non-zero terms are restricted to triangles of the family $\mathcal{T}_d$ of triangles cut by the discontinuity.

Due to the anisotropy assumption, each triangle of the mesh is near of an equilateral one, moreover, for those of $\mathcal{T}_d$, we note that:

- most efficient isotropic meshes will have a quasi-constant mesh size along the discontinuity. The optimal mesh is essentially a quasi-uniform strip covering the discontinuity and made of quasi-equilateral triangles.

- the position of the strip with respect to the discontinuity is also identified : the case where the discontinuity is close to a boundary of the strip maximizes the interpolation error and thus the intermediate position, with the discontinuity lying about the center of the strip is likely the optimal one.

Let $\Delta x$ be the length of a side of the above triangles, the square of the $L^2$ interpolation error on each element is proportional to the element area, that is to $(\Delta x)^2$, and the corresponding "covered" length of discontinuity is of the order of $\Delta x$.

- In order to cover the discontinuity over one unit length, about $k\ N$, triangles are necessary with $N = 1/(\Delta x)$, in other words, $k/(\Delta x)$ vertices are needed. And this will result in a error square as large as $\Delta x$. The error estimate is then $(\Delta x)^{1/2}$. In terms of numbers of vertices, the error for $N$ vertices is of the order of $(1/N)^{1/2}$, or $(1/N)^{1/d}$ which is exactly first-order convergence.

***Remark 4.2***
*In $L^p$ $(1 < p < \infty)$ the above argument gives a maximal order of $d/p$.*

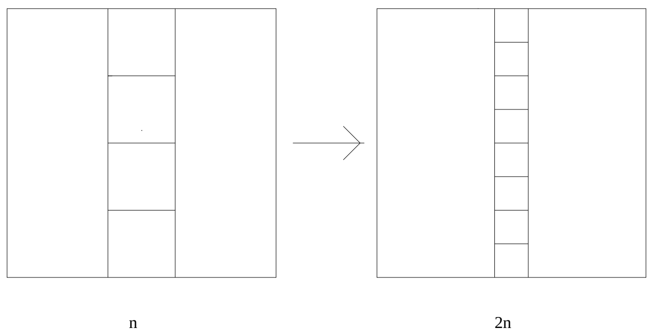


Figure 2: Isotropic adaptation for a (vertical) discontinuity. For dividing the $L^2$ by a factor $\sqrt{2}$, twice as many degrees of freedom are necessary. The figure presents the building of a new mesh by isotropic division when square elements are used.

## 4.3 A first 3D case

The above calculation can be transposed to a singularity born by a straight line in 3D. This can be for example a creek, in Solid Mechanics.

***Lemma 4.3*** *($d = 3$)*
*Under asumption of Lemma 3, and for a function $u$ defined on $\mathbb{R}^3$, having a singularity born by a straight line, the best order of accuracy for isotropic adaptive strategies is 3.*

Let $\Delta x$ be the length of the side of any tetrahedron. On each tetrahedron the square of the error in $L^2$ is of the order of $(\Delta x)^3$ for "covering" a section of the discontinuity of length

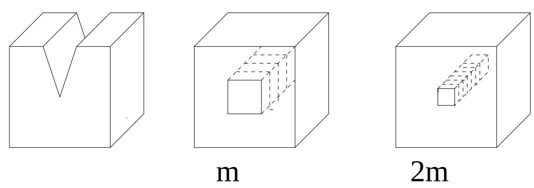


Figure 3: Generation of a new adapted mesh in the case of a creek.

$\Delta x$. For covering the whole discontinuity over one unit of length, the number of element needed will be of the order of $1/(\Delta x)$, or equivalently $1/(\Delta x)$ vertices/nodes. The square of the interpolation error in $L^2$ is of the order of $(\Delta x)^2$. Then we need $N$ nodes for an error of the order of $(\Delta x) = (1/N) = (1/N)^{3/d}$. This corresponds to third-order accuracy.

***Remark 4.3***
*In the case of a $\mathcal{P}^1$ interpolation the error would be of order 2 on regular region and thus the above estimate does not carry any extra limitation in accuracy.*

***Remark 4.4***
*Conversely, uniform refinement strategy would have required $N^3$ nodes for the same level of error, that is to say, only a first order accuracy would be obtained.*

***Remark 4.5***
*An example of creek function with this kind of singularity can be found in [2], where an isotropic adaptive strategy with high order interpolation reaches indeed this accuracy order.*

## 4.4 A limiting barrier for the isotropic 3D case

***Lemma 4.4** ($d = 3$)*
*Let $u$ be a function of three real variables, having a discontinuity born by a surface. Then the maximal order of convergence of an mesh adaptative isotropic $P_1$-interpolation method is $3/4$.*
*If the discontinuity concerns the gradient of $u$, the order is at best $3/2$.*

This time $1/(\Delta x)^2$ vertices are necessary to capture optimally the discontinuity and this would produce a square of error on element of the order of $(\Delta x)^3$. Thus the total error square will be in $(\Delta x)$ and the error in $\sqrt{\Delta x}$. Putting a number $N$ of vertices equal to $1/(\Delta x)^2$, we get an error in $(1/n)^{1/4} = ((1/n)^{3/4})^{1/3}$ which results in a maximal order of $3/4$.

In the case of a gradient discontinuity, the square of error on each element is of order $(\Delta x)^4$, the square of global error of order $(\Delta x)^2$, the error of order $\Delta x$, which results in a maximal order of $3/2$.

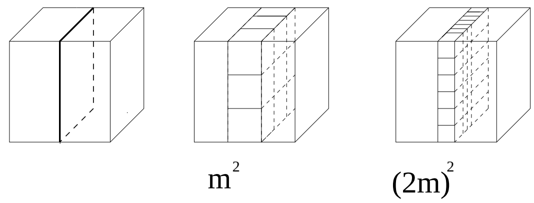


Figure 4: Generation of a new mesh by isotropic division in 3D, for a discontinuity born by a surface of $R^3$.

***Remark 4.6***
*Of course, the qualification condition for higher order accuracy is not that mesh adaptation be of anisotropic type, but be able to stretch mesh in any arbitrary direction.*

# 5 Numerical Illustration

The above theoretical may have an important practical impact. In particular, it is very easy to experience the above barriers by direct application of a mesh adaptative strategy.

This will be realized by using a modern mesh adaptation algorithm. It consists of a loop of successive mesh adaptation with a prescribed number of nodes. New meshes are triangulations generated according to metrics derived from an approximation of the Hessian (at previous mesh nodes).

The problem addressed is defined as follows. We want to interpolate as accurately as possible with the continuous piecewise $P_1$ operator and a unique adapted mesh of a square, two given discontinuous functions. The Heavyside functions $H_1$ et $H_2$ are discontinuous respectively on $x = 0$ for the first one, on $y = 0$ for the second one. For a prescribed number $n$ of vertices, we obtain a $L^2$ error $\mathcal{E}_n$. We compute this error for various mesh fineness $n$ and we measure the slope of the so-obtained convergence curve. It is compared to the above definition of convergence order.

## 5.1 Uniform refinement

In a first series of computations the triangulations are quasi uniform. The above theory predicts a convergence order bounded by $1/2$. The number of nodes of the meshes that have been applied cover a range from 60 to 20000. Figure 4 exhibits numerical convergence orders that are approximatively situated between $1/2$ and 1.

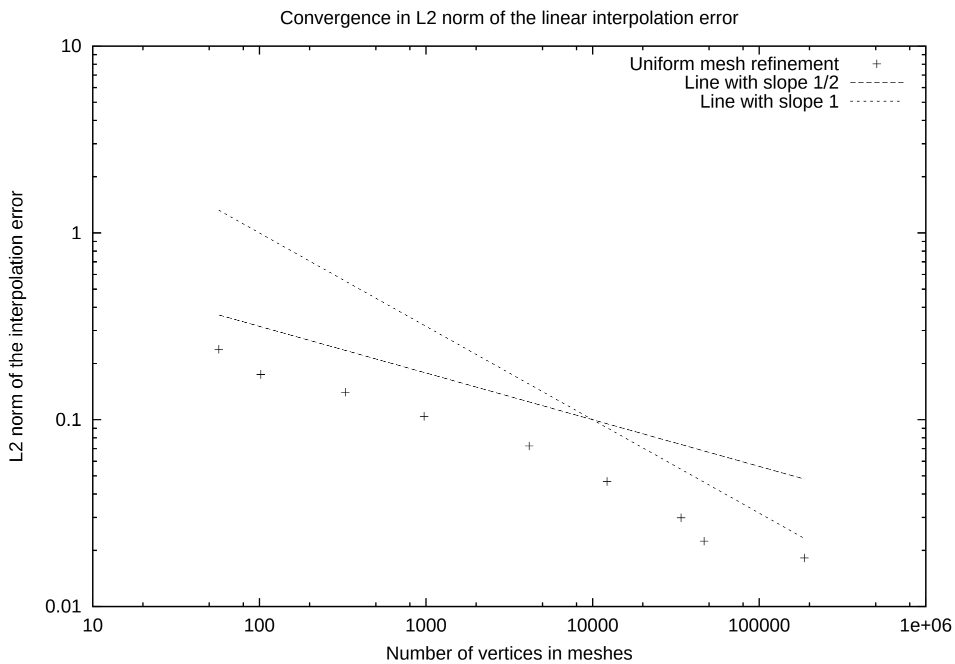


Figure 5: Convergence in $L^2$ norm of the interpolations of two Heavyside functions, with a uniform mesh refinement

## 5.2 Isotropic adaptation

A second series of calculations has been done with mesh adaptation. The **BAMG** adaptator ([3]) is used with a scalar metrics defined on any vertex of the current mesh, and chosen as the maximum of the absolute values of the eigenvalues of the local approximate Hessian of the two functions to interpolate. One result is obtained for a prescribed number of vertices when the adaptation loop produces successive meshes that are very close to each other.

A typical example of result of this loop is depicted in Figure 5. High mesh density is observed in the neighboring of the two discontinuities.

The numerical order of convergence can be observed from a series of computation with mesh fineness ranging from 600 to 200000 vertices. Interpolation error are much smaller than with the uniform meshes of same number of nodes. The order of convergence is now clearly close to 1.

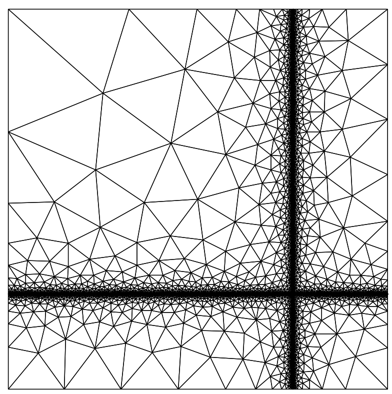

Figure 6: An example of isotropic adapted mesh for two Heavyside functions

## 5.3 Anisotropic adaptation

In the anisotropic adaptation, the metrics is a matrix, defined on any vertex, obtained from the discrete Hessian by replacing eigen values by their absolute values. The synthesis between the two functions to interpolate follows the method proposed in BAMG, computing an ellipse inside the two stretching ellipses, see [3]. Again the adaptation iteration loop is reiterated until the successive meshes do not show much variation. A typical example of adapted-converged mesh is presented in Figure 7.

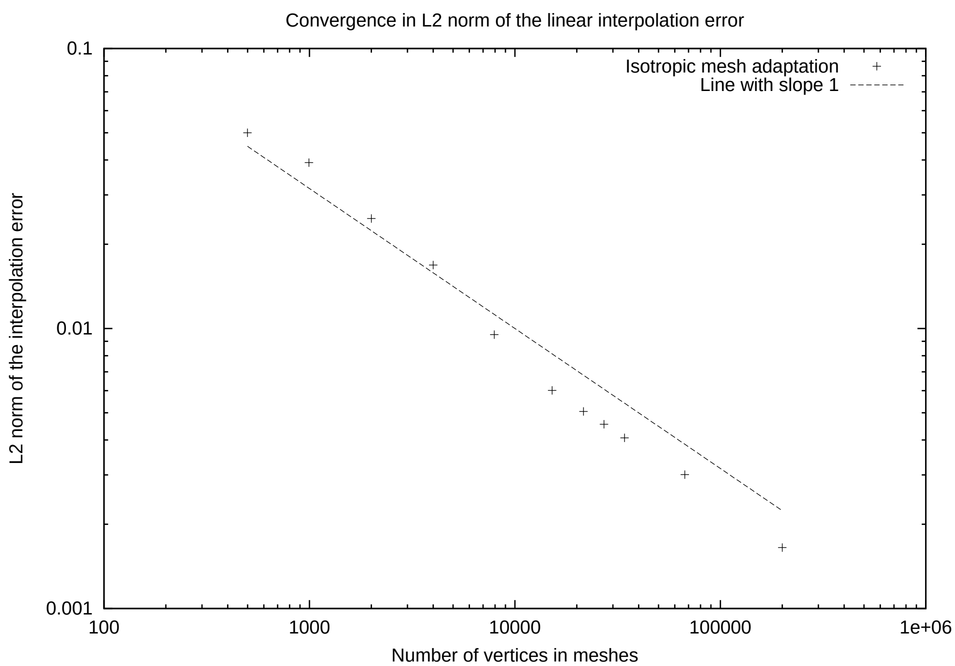


Figure 7: Convergence in $L^2$ norm of the $P_1$ interpolation of two Heavyside functions with an isotropic adaptation strategy

The numerical convergence is now measured with a series of meshes with fineness from 150 and 200000 vertices. Not only errors are much smaller than with isotropic adaptation, but also the convergence order is very close to 2.

***Remark 5.1***
*As already pointed out, if the functions to interpolate are regular, but with stiff gradients, then the metrics should distribute vertices between the regular zone and the discontinuity. Further, with an interpolator of higher order of accuracy (3 instead of 2,...) this distribution should operate more in favour of the singular region in order to get a global third order convergence.*

***Remark 5.2***
*The above example is a little particular since the functions are constant outside the discontinuities. All mesh vertices should concentrate near the discontinuities and not anywhere else. As a result, the convergence order is not limited by the order of the interpolator, and it could be better than 2 with the anisotropic adaptation. In the practical presented case, the order of convergence is limited by mesh regularity and progressivity, which lead to generate a medium-fine mesh in order to match coarse mesh regions and fine mesh ones.*

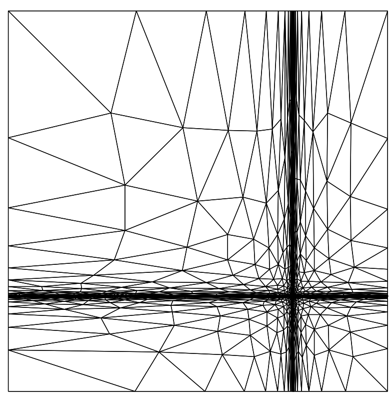

Figure 8: An example of isotropic adapted mesh for two Heavyside functions

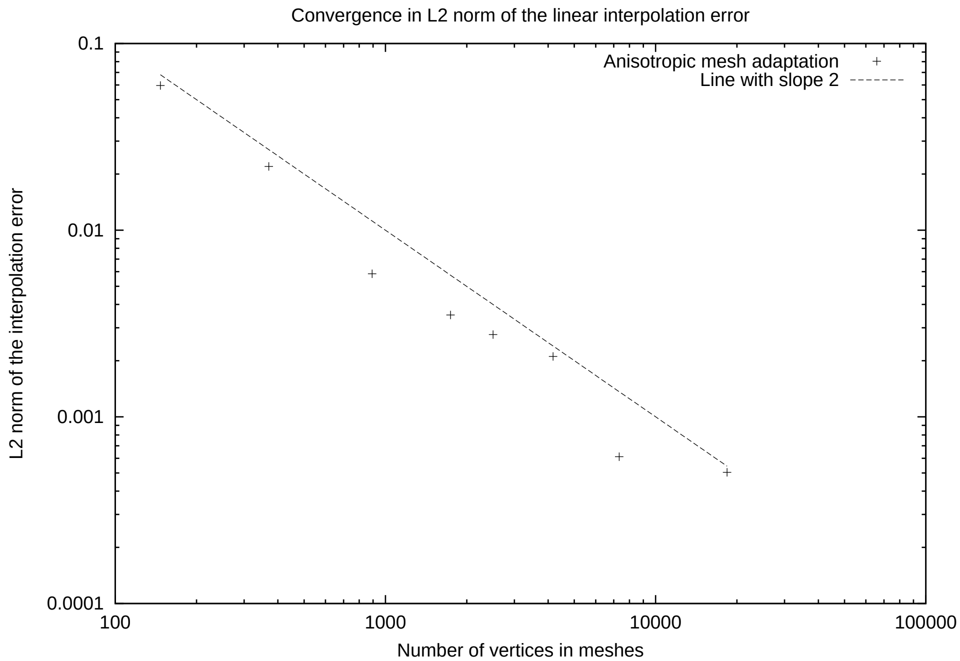


Figure 9: Convergence in $L^2$ norm of the $P_1$ interpolation of two Heavyside functions with an anisotropic adaptation strategy

# 6 Further remarks

In the general case of the approximation of PDE's, interpolation errors are generally involved and the above barriers should also apply.

It is possible to extend the present remarks to a singularity that lies into the **non-boundedness of the domain**. A typical case is the flow around an airfoil. In such a case it is evident that the domain cannot be meshed uniformly without an infinity of elements. If a truncation before infinity is applied in order to have a bounded mesh, then uniform refinement without enlarging the mesh is also an inefficient strategy since the error of domain truncation can become dominant. Then the only reasonable mesh refinement strategy should be also adaptative with respect to the behavior of the solution at farfield.

The **unsteady** case also sets problems of efficiency. Again the case of the approximation of a PDE cannot be solved with a better accuracy that the case on an interpolation and our discussion will concentrate on the interpolation case.

A preliminary remark is that unsteadiness increases of one unity the dimension of the approximation or interpolation domain. Then barriers are necessarily even more severe.

In fact the only strategy that might avoid a barrier is a "fully space-time adaptative anisotropic" strategy, which is hardly compatible with a discretisation with time levels. A "fully space-time adaptative isotropic" strategy would be limited to order 2/3.

If we consider strategies with time levels, it is useful to distinguish whether time step is adapted or not. Probably the best approach with time levels will use a different mesh, adapted, anisotropic at every time level. In that case, due to the uniform in time discretization, it will remains a barrier at some order.

The popular method of AMR has a different mesh, adapted at each time step, but the spatial adaptation is of isotropic type. It is less good than a "fully space-time adaptative isotropic" strategy and limited by a barrier even more severe than 2/3.

# 7 Conclusion

This study concentrates on the approximation of non-smooth functions. One of our arguments is that non-smooth functions encountered in physics have very small subregion of non-smoothness. Further, singularities are often born by rather regular manifolds. Of course the difficulty in approximation depends strongly on the nature of the singularity.

Our study is an attempt to evaluate the respective efficiency of various mesh-based approximation strategies for converging approximations to a non-smooth function.

Mesh refinement is the classical one. It is compared with various mesh adaptation strategies, using an isotropic mesh regeneration or anisotropic one.
All these methods are evaluated and compared from their performance in the approximation of discontinuous, piecewise regular functions.
Our study relies on theoretical remarks and counter-examples. It is completed by a few concrete numerical calculations.

The output of the study is summed up as follows:
- Uniform refinement is definitively an inefficient approach for discontinuous functions.
- Isotropic adaptative strategies are unable in 2D and in 3D cases to show an asymptotical accuracy order better than first order (order 3/4 for 3D) when they are applied to discontinuities beared by curves in 2D (surfaces in 3D).

This remark explains in particular the disappointing results obtained when isotropic refinement is applied to the capturing of shock surfaces around 3D aircrafts. As far as we know, a theoretical quantification of these difficulties was not proposed up to now.

In the elliptic case of Computational Structural Mechanics, discontinuities in the medium properties provoke only derivative dicontinuities. However, their computation with an isotropic adaptative strategy cannot be better than of order 3/2 in 3D.

In order to (try to) escape these limitations, it seems essential to apply anisotropic mesh adaptive methods, especially in the 3D case.

Several anisotropic mesh adataption methods are proposed in the litterature and two good examples can be found in [5] and [3]. They allow a much weaker refinement in the direction that are tangent to the singularity. Then the above barrier dœs not apply and it is very probable, but this is yet almost a conjecture, that any order of accuracy can be obtained with good anisotropic mesh adaptative strategies.

To synthetize, obtaining a high order of convergence towards a function with a local singularity will depend of the following three factors:
- the accuracy order $\alpha$ of the interpolation (or approximation),
- the choice of an adequate sensor concentrating the mesh in an anisotropic manner, with a strength adequately tuned with $\alpha$ (stronger with a higher $\alpha$).
- the ability of the mesh regenerator to follow the indication of the sensor, and in particular to apply mesh density with very sudden and strong variations.

Many bricks of the building are needed in order to extend rigorously the proposed theory to numerical approximation of PDE. We show, however, in [7],[6], that a strategy relying on the same mesh regenerator BAMG as in our example effectively provide second order accuracy for non-regular test cases in Computational Fluid Dynamics (for which uniform refinement fails).

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